Superconvergence of Discontinuous Galerkin Method for Scalar Nonlinear Hyperbolic Equations

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In this paper, we study the superconvergence behavior of the semi-discrete discontinuous Galerkin (DG) method for scalar nonlinear hyperbolic equations in one spatial dimension. Superconvergence results for problems with fixed and alternating wind directions are established. On the one hand, we prove that, if the wind direction is fixed (i.e., the derivative of the flux function is bounded away from zero), both the cell average error and numerical flux error at cell interfaces converge at a rate of $2k+1$ when upwind fluxes and piecewise polynomials of degree k are used. Moreover, we also prove that the function value approximation of the DG solution is superconvergent at interior right Radau points, and the derivative value approximation is superconvergent at interior left Radau points, with an order of $k+2$ and $k+1$, respectively. As a byproduct, we show a $k+2$-th order superconvergence of the DG solution towards the Gauss-Radau projection of the exact solution. On the other hand, superconvergence results for problems with alternating wind directions (i.e., the derivative of the flux function either changes sign or otherwise achieves the value zero in the domain) are also established. To be more precise, we first prove that the DG flux function is superconvergent towards a particular flux function of the exact solution, with an order of $k+2$, when Godunov fluxes are used. We then prove that the highest superconvergence rate of the DG solution itself is $k+\frac{3}{2}$ when sonic points (i.e., the derivative of the flux function achieves zero) appear in the computational domain. As byproducts, we obtain superconvergence properties for the DG solution and the DG flux function at special points and for cell average. Numerical experiments demonstrate that most of our results are optimal, i.e., the superconvergence rates are sharp.